A lower bound on the order of counterexamples to Seymour's Second Neighborhood Conjecture

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Abstract

We show that any counterexample to Seymour's Second Neighborhood Conjecture on n vertices with minimum out-degree δ satisfies $n \geq 2\delta + 3$. Combined with the recent result of Sadhukhan, Sandeep and Sen that the conjecture holds for minimum out-degree at most 7, this implies that any counterexample has at least 19 vertices.

Let D be an oriented graph: a finite directed graph with no loops and no digons. For $v \in V(D)$, let $N^+(v)$ denote the out-neighborhood of v, let $d^+(v) = |N^+(v)|$, and let

$$N^{++}(v) \;=\; \left\{\, w \in V(D) \setminus (N^+(v) \cup \{v\}) \;:\; u \to w \text{ for some } u \in N^+(v) \,\right\}$$

denote the second out-neighborhood. Seymour's Second Neighborhood Conjecture, first published by Dean and Latka [1], asserts that every oriented graph contains a vertex v with $|N^{++}(v)| \geq d^+(v)$. A *counterexample* is an oriented graph in which $|N^{++}(v)| < d^+(v)$ for every vertex v; note that every vertex of a counterexample has out-degree at least one.

Theorem 1. *Let D be a counterexample to the Second Neighborhood Conjecture on n vertices with minimum out-degree δ. Then $n \geq 2\delta + 3$.*

Proof. Since D is oriented, $n\delta \leq \sum_v d^+(v) = |E(D)| \leq \binom{n}{2}$, so $n \geq 2\delta + 1$.

Case $n = 2\delta + 1$. Equality holds throughout, so D is a tournament. Tournaments satisfy the conjecture by Fisher's theorem [3], so D is not a counterexample.

Case $n = 2\delta + 2$. For $v \in V(D)$ set $e_v = d^+(v) - \delta \geq 0$, let t_v be the number of non-neighbors of v in the underlying undirected graph, and set $r_v = e_v + t_v$. Let M be the number of non-adjacent unordered pairs. Counting arcs,

$$\sum_v d^+(v) \;=\; |E(D)| \;=\; \binom{2\delta+2}{2} - M, \qquad \text{hence} \qquad \sum_v e_v = \delta + 1 - M.$$

Since $\sum_v t_v = 2M$, we get $R := \sum_v r_v = \delta + 1 + M \geq 1$.

Let $Z = \{v : d^+(v) = \delta\}$. At most $\sum_v e_v = \delta + 1 - M$ vertices have $e_v \geq 1$, so

$$|Z| \;\geq\; (2\delta+2) - (\delta + 1 - M) \;=\; \delta + 1 + M \;=\; R.$$

Each $v \in Z$ satisfies $|N^{++}(v)| \leq \delta - 1$, so among the $2\delta + 1$ vertices other than v, at least $(2\delta+1) - \delta - (\delta - 1) = 2$ lie outside $\{v\} \cup N^+(v) \cup N^{++}(v)$. Let

$$P \;=\; \left\{\, (v, w) \;:\; v \in Z,\; w \notin \{v\} \cup N^+(v) \cup N^{++}(v) \,\right\}, \qquad \text{so} \qquad |P| \;\geq\; 2|Z| \;\geq\; 2R. \tag{1}$$

Now fix $w \in V(D)$ and let $B_w = \{v \in Z : (v, w) \in P\}$, $b_w = |B_w|$, and

$$A_w \;=\; N^+(w) \cup \{\text{non-neighbors of } w\}, \qquad |A_w| \;=\; d^+(w) + t_w \;=\; \delta + r_w.$$

Let $v \in B_w$. Then $v \not\to w$, so either $w \to v$ or $\{v, w\}$ is a non-edge; in both cases $v \in A_w$. Moreover, for every $u \in N^+(v)$ we have $u \neq w$ (since $w \notin N^+(v)$) and $u \not\to w$ (otherwise $w \in N^{++}(v)$); hence $u \in A_w$. Therefore

$$v \in A_w \qquad \text{and} \qquad N^+(v) \subseteq A_w \setminus \{v\}.$$

Since $|N^+(v)| = \delta$ and $|A_w \setminus \{v\}| = \delta + r_w - 1$, the vertex v out-dominates all but at most $r_w - 1$ vertices of $A_w \setminus \{v\}$. As $B_w \subseteq A_w$, in the induced oriented graph $D[B_w]$ every vertex has out-degree at least $b_w - r_w$.

We claim that $b_w \geq 1$ implies $r_w \geq 1$ and $b_w \leq 2r_w - 1$. Indeed, if $b_w > r_w$, counting arcs in $D[B_w]$ gives $b_w(b_w - r_w) \leq \binom{b_w}{2}$, hence $b_w \leq 2r_w - 1$; and if $1 \leq b_w \leq r_w$, then $r_w \geq 1$ and $b_w \leq r_w \leq 2r_w - 1$ directly. Summing over w,

$$|P| \;=\; \sum_w b_w \;\leq\; \sum_{w\,:\,r_w \geq 1} (2r_w - 1) \;=\; 2R - \bigl|\{w : r_w \geq 1\}\bigr| \;\leq\; 2R - 1, \tag{2}$$

using $R \geq 1$ in the last step. Inequalities (1) and (2) contradict each other, so $n \neq 2\delta + 2$. $\square$

Corollary 2. *Any counterexample to the Second Neighborhood Conjecture has at least* 19 *vertices.*

Proof. Sadhukhan, Sandeep and Sen [5] prove the conjecture for oriented graphs of minimum out-degree at most 7, extending the threshold of Kaneko and Locke [4]. Hence any counterexample has $\delta \geq 8$, and Theorem 1 gives $n \geq 2 \cdot 8 + 3 = 19$. $\square$

Remark. *Seacrest [6] showed that if a counterexample with minimum out-degree δ exists, then one exists on at most $\binom{\delta+1}{2}$ vertices. Consequently, if some counterexample has minimum out-degree exactly* 8*, then a counterexample D exists with $19 \leq |V(D)| \leq 36$. Moreover, Espuny Díaz, Girão, Granet and Kronenberg [2] showed that a vertex-minimal counterexample satisfies $\delta^+ > \sqrt{n}$; combined with Theorem 1, a vertex-minimal counterexample on* 19 *vertices would necessarily have $\delta^+ = 8$ exactly.*

Acknowledgment

The argument for the case $n = 2\delta + 2$ was produced by a large language model; the author verified it by hand. This note records the bound and asks whether it, or an equivalent formulation, is already known.

References

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